

Integrated and Differentiated Sequence Spaces and Weighted Mean

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Abstract

The purpose of this paper is twofold. Firstly, the new matrix domains are constructed with the new infinite matrices and some properties are investigated. Furthermore, dual spaces of new matrix domains are computed and matrix transformations are characterized. Secondly, examples between new spaces with classical sequence spaces and sequence spaces which are derived by an infinite matrix are given in the table form.

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1 Introduction

It is well known that, the ω denotes the family of all real (or complex)-valued sequences. ω is a linear space and each linear subspace of ω (with the included addition and scalar multiplication) is called a *sequence space* such as the spaces c , c_0 and ℓ_∞ , where c , c_0 and ℓ_∞ denote the set of all convergent sequences in fields \mathbb{R} or \mathbb{C} , the set of all null sequences and the set of all bounded sequences, respectively. It is clear that the sets c , c_0 and ℓ_∞ are the subspaces of the ω . Thus, c , c_0 and ℓ_∞ equipped with a vector space structure, form a sequence space. By bs and cs , we define the spaces of all bounded and convergent series, respectively.

A *coordinate space* (or K -space) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space X with a linear topology is called a K -space provided each of the maps $p_i : X \rightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K -space is called an FK -space provided X is a complete linear metric space. An FK -space whose topology is normable is called a BK -space. If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is unique sequence of scalars (α_n) such that

$$\lim_{n \rightarrow \infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \dots + \alpha_n b_n)\| = 0$$

then (b_n) is called *Schauder basis* for X . The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) , and written as $x = \sum \alpha_k b_k$. An *FK-space* X is said to have *AK* property, if $\phi \subset X$ and $\{e^k\}$ is a basis for X , where e^k is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$ and $\phi = \text{span}\{e^k\}$, the set of all finitely non-zero sequences.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} and $x = (x_k) \in \omega$, where $k, n \in \mathbb{N}$. Then the sequence Ax is called as the A -transform of x defined by the usual matrix product. Hence, we transform the sequence x into the sequence $Ax = \{(Ax)_n\}$ where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (1)$$

for each $n \in \mathbb{N}$, provided the series on the right hand side of (1) converges for each $n \in \mathbb{N}$. Let X and Y be two sequence spaces. If Ax exists and is in Y for every sequence $x = (x_k) \in X$, then we say that A defines a matrix mapping from X into Y , and we denote it by writing $A : X \rightarrow Y$ if and only if the series on the right hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$ for all $x \in X$. A sequence x is said to be A -summable to l if Ax converges to l which is called the A -limit of x . Let X be a sequence space and A be an infinite matrix. The sequence space

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} \quad (2)$$

is called the domain of A in X which is a sequence space.

We write \mathcal{U} for the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. For $u \in \mathcal{U}$, let $1/u = (1/u_k)$. Let $u, w \in \mathcal{U}$. Now, we define the *generalized weighted mean* or *factorable matrix* $G(u, w) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n w_k & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$; where u_n depends only on n and w_k only on k .

By \mathcal{F} , we will denote the collection of all finite subsets on \mathbb{N} . For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞ . Also we use the convention that any term with negative subscript is equal to zero.

2 New Integrated and Differentiated Spaces

In this section, we will give new spaces defined by a weighted mean.

The concepts of integrated and differentiated sequence spaces was firstly used by Goes and Goes [7] as $\int X = \{x = (x_k) \in \omega : (kx_k) \in X\}$ and $d(X) = \{x = (x_k) \in \omega : (k^{-1}x_k) \in X\}$. Malkowsky and Savaş [19] have defined the sequence space $Z = (u, v; X)$, which consists of all sequences whose $G(u, v)$ -transforms are in $X \in \{\ell_\infty, c, c_0, \ell_p\}$, where $u, w \in \mathcal{U}$. Paranormed sequence spaces derived by weighted mean are studied in [3]. Altay and Başar [4] constructed the new paranormed sequence spaces $\ell(u, v; p)$. Şimsek et al. [21] have introduced a modular structure of the sequence spaces defined by Altay and Başar [4] and studied Kadec-Klee and uniform Opial properties of this sequence space on Köthe sequence spaces. In [22], using the generalized weighted mean, new difference sequence spaces are defined. Kirişci [11] have defined the almost sequence spaces with generalized weighted mean and in [12], studied some properties of new almost sequence spaces derived by generalized weighted mean. Structural properties of the bv space are studied by Cesaro mean, generalized weighted mean and Riesz mean, in [13]. Following the Goes and Goes [7], Kirişci [14] have studied the integrated and differentiated sequence spaces and defined the Riesz type integrated and differentiated sequence spaces, in [15].

We define the new matrices $\Gamma = (\gamma_{nk})$ and $\Sigma = (\sigma_{nk})$ by

$$\gamma_{nk} = \begin{cases} ku_n(w_k - w_{k+1}) & , & (k < n) \\ nu_nw_n & , & (n = k) \\ 0 & , & (k > n) \end{cases} \quad (3)$$

$$\sigma_{nk} = \begin{cases} \frac{1}{k}u_n(w_k - w_{k+1}) & , & (k < n) \\ \frac{u_nw_n}{n} & , & (n = k) \\ 0 & , & (k > n) \end{cases} \quad (4)$$

for all $k, n \in \mathbb{N}$.

Let $u, w \in \mathcal{U}$. The new integrated spaces defined by

$$\int bv(u, w) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^n u_n w_k \Delta(kx_k) < \infty \right\}$$

and the new differentiated spaces defined by

$$d(bv(u, w)) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^n u_n w_k \Delta(k^{-1}x_k) < \infty \right\}.$$

Consider the notation (2) and the matrices (3), (4). From here, we can re-define the spaces $\int bv(u, w)$ and $d(bv(u, w))$ by

$$(\ell_1)_\Gamma = \int bv(u, w) \quad (5)$$

and

$$(\ell_1)_\Sigma = d(bv(u, w)). \quad (6)$$

Let $x = (x_k) \in \int bv(u, w)$. The Γ -transform of a sequence $x = (x_k)$ is defined by

$$y_n = \sum_{k=1}^{n-1} k u_n (w_k - w_{k+1}) x_k + n u_n w_n x_n \quad (7)$$

where Γ is defined by (3). Let $x = (x_k) \in d(bv(u, w))$. The Σ -transform of a sequence $x = (x_k)$ is defined by

$$y_n = \sum_{k=1}^{n-1} \frac{1}{k} u_n (w_k - w_{k+1}) x_k + \frac{1}{n} u_n w_n x_n \quad (8)$$

where Σ is defined by (4).

Theorem 2.1. *The integrated and differentiated sequence spaces derived by weighted mean are norm isomorphic to the absolute summable sequence space.*

Proof. We must show that a linear bijection between the integrated sequence space derived by weighted mean and the absolute summable sequence space exists. Consider the transformation f_Γ defined, with the notation (7), from $\int bv(u, w)$ to ℓ_1 by $x \mapsto y = f_\Gamma x$. The linearity of f_Γ is clear. Also, it is trivial that $x = \theta$ whenever $f_\Gamma x = \theta$ and therefore, f_Γ is injective.

Let $y \in \ell_1$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=1}^{k-1} \frac{1}{k} \frac{1}{u_j} \left(\frac{1}{w_j} - \frac{1}{w_{j+1}} \right) y_j + \frac{y_k}{k \cdot u_k w_k}.$$

Then

$$\|x\|_{\int bv(u, w)} = \sum_k \left| \sum_{j=1}^{k-1} j u_k (w_j - w_{j+1}) x_j + n u_n w_n x_n \right| = \sum_k |y_k| = \|y\|_{\ell_1} < \infty.$$

Then, we have that $x \in \int bv(u, w)$. So, f_Γ is surjective and norm preserving. Hence f_Γ is a linear bijection. It shown us that the space $\int bv(u, w)$ is norm isomorphic to ℓ_1 .

As similar, using the notation (8), we can define the transformation f_Σ from $d(bv(u, w))$ and ℓ_1 by $x \mapsto y = f_\Sigma x$. If we choose the sequence $x = (x_k)$ by

$$x_k = \sum_{j=1}^{k-1} k \frac{1}{u_j} \left(\frac{1}{w_j} - \frac{1}{w_{j+1}} \right) y_j + \frac{k \cdot y_k}{u_k w_k}$$

while $y \in \ell_1$, then we obtain the space $d(bv(u, w))$ is norm isomorphic to ℓ_1 with the norm $\|x\|_{d(bv(u, w))}$. \square

Since $\int bv(u, w) = [\ell_1]_\Gamma$ and $d(bv(u, w)) = [\ell_1]_\Sigma$ holds, ℓ_1 is a BK -space with the norm $\|x\|_{\ell_1}$ and the matrices Γ and Σ are triangle matrix, then Theorem 4.3.2 of Wilansky[23] gives the fact that the integrated and differentiated sequence spaces derived by weighted mean are BK -space. Therefore, there is no need for detailed proof of the following theorem.

Theorem 2.2. *The spaces $\int bv(u, w)$ and $d(bv(u, w))$ are BK -space with the norms $\|x\|_{\int bv(u, w)} = \|x\|_{\ell_1(\Gamma)}$ and $\|x\|_{d(bv(u, w))} = \|x\|_{\ell_1(\Sigma)}$, respectively.*

Theorem 2.3. *The differentiated sequence space derived by weighted mean has AK -property.*

Proof. Let $x = (x_k) \in d(bv(u, w))$ and $x^{[n]} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$. Hence,

$$x - x^{[n]} = \{0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots\} \Rightarrow \|x - x^{[n]}\|_{d(bv(u, w))} = \|0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots\|$$

and since $x \in d(bv(u, w))$,

$$\begin{aligned} \|x - x^{[n]}\|_{d(bv(u, w))} &= \sum_{k \geq n+1} \left| \frac{1}{k} u_n (w_k - w_{k+1}) x_k + \frac{1}{n} u_n w_n x_n \right| \rightarrow 0 \text{ as } n \rightarrow \infty \\ &\Rightarrow \lim_{n \rightarrow \infty} \|x - x^{[n]}\|_{d(bv(u, w))} = 0 \Rightarrow x^{[n]} \rightarrow x \text{ as } n \rightarrow \infty \text{ in } d(bv(u, w)). \end{aligned}$$

Then the space $d(bv(u, w))$ has AK -property. \square

Because of the isomorphisms f_Γ and f_Σ , defined in the proof of Theorem 2.1, are onto the inverse image of the basis $\{e^{(k)}\}_{k \in \mathbb{N}}$ of the space ℓ_1 is the basis of the spaces $\int bv(u, w)$ and $d(bv(u, w))$. Therefore, we can give following theorems for Schauder basis of new sequence spaces :

Theorem 2.4. *Define a sequence $s^{(k)} = \{s_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space $\int bv(u, w)$ for every fixed $k \in \mathbb{N}$ by*

$$s_n^{(k)} = \begin{cases} \frac{1}{n} \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) & , \quad (1 < k < n) \\ \frac{1}{n u_n w_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases}$$

Therefore, the sequence $\{s^{(k)}\}_{k \in \mathbb{N}}$ is a basis for the space $\int bv(u, w)$ and any $x \in \int bv(u, w)$ has a unique representation of the form

$$x = \sum_k (\Gamma x)_k s^{(k)} \quad (9)$$

Proof. Let $e^{(k)}$ be a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$. We know that

$$\Gamma s^{(k)}(q) = e^{(k)} \in \ell_1 \quad (10)$$

for all $k \in \mathbb{N}$. Then, we have $\{s^{(k)}(q)\} \subset \int bv(u, w)$.

We take $x \in \int bv(u, w)$. Then, we put,

$$x^{[m]} = \sum_{k=1}^m (\Gamma x)_k(q) s^{(k)}(q), \quad (11)$$

for every positive integer m . Then, we have

$$\Gamma x^{[m]} = \sum_{k=1}^m (\Gamma x)_k(q) \Gamma s^{(k)}(q) = \sum_{k=1}^m (\Gamma x)_k(q) e^{(k)}$$

and

$$\left(\Gamma(x - x^{[m]}) \right)_i = \begin{cases} 0 & , \quad (1 \leq i < m) \\ (\Gamma x)_i & , \quad (i > m) \end{cases}$$

by applying Γ to (11) with (10), for $i, m \in \mathbb{N}$. For $\varepsilon > 0$, there exists an integer m_0 such that

$$\left[\sum_{i=m}^{\infty} |(\Gamma x)_i| \right] < \varepsilon/2$$

for all $m \geq m_0$. Hence,

$$\|x - x^{[m]}\|_{\int bv(u, w)} =$$

for all $m \geq m_0$. Therefore, $x \in \int bv(u, w)$ is represented as in (9), as we desired. \square

Theorem 2.5. Define a sequence $t^{(k)} = \{t_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space $d(bv(u, w))$ for every fixed $k \in \mathbb{N}$ by

$$t_n^{(k)} = \begin{cases} n \left(\frac{1}{u_k w_k} - \frac{1}{u_k u_{k+1}} \right) & , \quad (1 < k < n) \\ \frac{n}{u_n w_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases}$$

Therefore, the sequence $\{t^{(k)}\}_{k \in \mathbb{N}}$ is a basis for the space $d(bv(u, w))$ and any $x \in d(bv(u, w))$ has a unique representation of the form

$$x = \sum_k (\Sigma x)_k t^{(k)}.$$

Remark. It is well known that every Banach space X with a Schauder basis is separable.

From Theorem 2.4, Theorem 2.5 and Remark, we can give following corollary:

Corollary 2.6. The spaces $\int bv(u, w)$ and $d(bv(u, w))$ are separable.

3 Dual Spaces

If $X, Y \subset \omega$ and z any sequence, we can write $z^{-1} * X = \{x = (x_k) \in \omega : xz \in X\}$ and $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y$. If we choose $Y = cs, bs$, then we obtain the β -, γ -duals of X , respectively as

$$\begin{aligned} X^\beta &= M(X, cs) = \{z = (z_k) \in \omega : zx = (z_k x_k) \in cs \text{ for all } x \in X\} \\ X^\gamma &= M(X, bs) = \{z = (z_k) \in \omega : zx = (z_k x_k) \in bs \text{ for all } x \in X\}. \end{aligned}$$

Let $A = (a_{nk})$ be an infinite matrix. Now we give some conditions:

$$\sup_{k, n \in \mathbb{N}} |a_{nk}| < \infty, \quad (12)$$

$$\lim_{n \rightarrow \infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \quad (13)$$

$$\sup_{k \in \mathbb{N}} \sum_n |a_{nk}| < \infty \quad (14)$$

$$\sup_{k, m \in \mathbb{N}} \left| \sum_{n=0}^m a_{nk} \right| < \infty, \quad (15)$$

$$\sum_n a_{nk} \text{ convergent for each } k \in \mathbb{N} \quad (16)$$

$$\sum_n a_{nk} = 0 \text{ for each } k \in \mathbb{N} \quad (17)$$

Lemma 3.1. *For the characterization of the class $(X : Y)$ with $X = \{\ell_1\}$ and $Y = \{\ell_\infty, c, \ell_1\}$, we can give the necessary and sufficient conditions from Table 1, where*

1. (12)	2. (12), (13)	3. (14)	4. (15)	5. (15), (16)	6. (15), (17)
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To \rightarrow	ℓ_∞	c	ℓ_1	bs	cs	c_0s
From \downarrow						
ℓ_1	1.	2.	3.	4.	5.	6.

Table 1

Theorem 3.2. *We define the matrix $E = (e_{nk})$ as*

$$e_{nk} = \begin{cases} \frac{1}{n} \frac{1}{u_k} \left(\frac{1}{w_k} - \frac{1}{w_{k+1}} \right) a_n & , \quad (1 \leq k < n) \\ \frac{a_n}{nu_n w_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases} \quad (18)$$

for all $k, n \in \mathbb{N}$, where $u, w \in \mathcal{U}$, $a = (a_k) \in \omega$. The α -dual of the space $\int bv(u, w)$ is the set

$$d_1 = \left\{ a = (a_k) \in \omega : \sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} e_{nk} \right| < \infty \right\}$$

Proof. Let $a = (a_k) \in \omega$. We can easily derive that with the notation (7) that

$$a_n x_n = \sum_{k=1}^n \frac{1}{n} \frac{1}{u_k} \left(\frac{1}{w_k} - \frac{1}{w_{k+1}} \right) a_n y_k + \frac{a_n y_n}{n u_n w_n} = \sum_{k=1}^n e_{nk} y_k = (Ey)_n \quad (19)$$

for all $k, n \in \mathbb{N}$, where $E = (e_{nk})$ is defined by (18). It follows from (19) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \int bv(u, w)$ if and only if $Ey \in \ell_1$ whenever $y \in \ell_1$. We obtain that $a \in [\int bv(u, w)]^\alpha$ whenever $x \in \int bv(u, w)$ if and only if $E \in (\ell_1 : \ell_1)$. Therefore, we get by Lemma 3.1 with E instead of A that $a \in [\int bv(u, w)]^\alpha$ if and only if $\sup_{k \in \mathbb{N}} \sum_n |e_{nk}| < \infty$. This gives us the result that $[\int bv(u, w)]^\alpha = d_1$. \square

Theorem 3.3. *We define the matrix $F = (f_{nk})$ as*

$$f_{nk} = \begin{cases} n \frac{1}{u_k} \left(\frac{1}{w_k} - \frac{1}{w_{k+1}} \right) a_n & , \quad (1 \leq k < n) \\ \frac{n a_n}{u_n w_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases} \quad (20)$$

for all $k, n \in \mathbb{N}$, where $u, w \in \mathcal{U}, a = (a_k) \in \omega$. The α -dual of the space $d(bv(u, w))$ is the set

$$d_2 = \left\{ a = (a_k) \in \omega : \sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} f_{nk} \right| < \infty \right\}$$

Theorem 3.4. *Let $u, w \in \mathcal{U}$ for all $k, n \in \mathbb{N}$. Then the β -dual of the space $\int bv(u, w)$ is $d_3 \cap cs$, where*

$$d_3 = \left\{ a = (a_k) \in \omega : \sum_{k=1}^n \left| \frac{1}{k} \frac{a_k}{u_k w_k} + \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) \sum_{j=k+1}^n \frac{1}{j} a_j \right| < \infty \right\}$$

Proof. Consider the equation

$$\begin{aligned} \sum_{k=1}^n a_k x_k &= \sum_{k=1}^n a_k \left[\sum_{j=1}^{k-1} \frac{1}{k} \left(\frac{1}{u_j w_j} - \frac{1}{u_j w_{j+1}} \right) y_j + \frac{y_k}{k u_k w_k} \right] \\ &= \sum_{k=1}^n \left| \frac{1}{k} \frac{a_k y_k}{u_k w_k} + \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) y_k \sum_{j=k+1}^n \frac{1}{j} a_j \right| = (H_n y) \end{aligned} \quad (21)$$

for all $n \in \mathbb{N}$, where the matrix $H = (h_{nk})$ is defined by

$$h_{nk} = \begin{cases} \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) \sum_{j=k+1}^n \frac{1}{j} a_j & , \quad (k > n) \\ \frac{1}{n} \frac{a_n}{u_n w_n} & , \quad (n = k) \\ 0 & , \quad (k < n) \end{cases} \quad (22)$$

for all $k, n \in \mathbb{N}$. Therefore, we deduce from Lemma 3.1 with (21) that $ax = (a_n x_n) \in cs$ whenever $x \in \int bv(u, w)$ if and only if $Hy \in c$ whenever $y \in \ell_1$. From (12) and (13), we have

$$\lim_n h_{nk} = \alpha_k \quad \text{and} \quad \sup_k \sum_n |h_{nk}| < \infty$$

which shows that $[\int bv(u, w)]^\beta = d_3 \cap cs$. \square

Theorem 3.5. $[\int bv(u, w)]^\gamma = d_3$.

Proof. We obtain from Lemma 3.1 with (21) that $ax = (a_n x_n) \in bs$ whenever $x \in \int bv(u, w)$ if and only if $Hy \in \ell_\infty$ whenever $y \in \ell_1$. Then, we see from (12) that $[\int bv(u, w)]^\gamma = d_3$. \square

Theorem 3.6. The β -dual of the space $d(bv(u, w))$ is $d_4 \cap cs$, where

$$d_4 = \left\{ a = (a_k) \in \omega : \sum_{k=1}^n \left| \frac{k \cdot a_k}{u_k w_k} + \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) \sum_{j=k+1}^n j \cdot a_j \right| < \infty \right\}$$

Theorem 3.7. $[d(bv(u, w))]^\gamma = d_4$.

4 Matrix transformations

We shall write for brevity that

$$\begin{aligned} \bar{a}_{nk} &= \sum_{j=1}^n \left| \frac{1}{k} \frac{a_{nk}}{u_k w_k} + \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) \sum_{j=k+1}^n \frac{1}{j} a_{nj} \right|, \\ \tilde{a}_{nk} &= \sum_{j=1}^n \left| \frac{k \cdot a_{nk}}{u_k w_k} + \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) \sum_{j=k+1}^n j \cdot a_{nj} \right|, \\ \bar{b}_{nk} &= \sum_{j=1}^{n-1} j \cdot u_n (w_j - w_{j+1}) a_{jk} + n \cdot u_n w_n a_{nk}, \\ \tilde{b}_{nk} &= \sum_{j=1}^{n-1} \frac{1}{j} \cdot u_n (w_j - w_{j+1}) a_{jk} + \frac{1}{n} \cdot u_n w_n a_{nk} \end{aligned}$$

for all $k, n \in \mathbb{N}$.

Theorem 4.1. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation

$$a_{nk} = \sum_{j=k}^{\infty} j \cdot (w_k - w_{k+1}) u_j b_{nj} \quad \text{or} \quad b_{nk} = \bar{a}_{nk} \quad (23)$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then $A \in (\int bv(u, w) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv(u, w)\}^\beta$ for all $n \in \mathbb{N}$ and $B \in (\ell_1 : Y)$.

Proof. Let Y be any given sequence. Suppose that (23) holds between the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$, and take into account that the spaces $\int bv(u, w)$ and ℓ_1 are linearly isomorphic.

Let $A \in (\int bv(u, w) : Y)$ and take any $y = (y_k) \in \ell_1$. Then $B\Gamma$ exists and $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv(u, w)\}^\beta$ which yields that (23) is necessary and $\{b_{nk}\}_{k \in \mathbb{N}} \in \ell_1^\beta$ for each $n \in \mathbb{N}$. Hence, By exists for each $y \in \ell_1$ and thus by letting $m \rightarrow \infty$ in the equality

$$\sum_{k=1}^m a_{nk} x_k = \sum_{k=1}^m \left| \frac{1}{k} \frac{a_{nk} y_k}{u_k w_k} + \left(\frac{1}{u_k w_k} - \frac{1}{u_k w_{k+1}} \right) y_k \sum_{j=k+1}^m \frac{1}{j} a_{nj} \right| \quad \text{for all } m, n \in \mathbb{N}$$

we obtain that $Ax = By$ which leads us to the consequence $B \in (\ell_1 : Y)$.

Conversely, let $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv(u, w)\}^\beta$ for each $n \in \mathbb{N}$ and $B \in (\ell_1 : Y)$, and take any $x = (x_k) \in \int bv(u, w)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=1}^m b_{nk} y_k = \sum_{k=1}^m a_{nk} x_k \quad \text{for all } m, n \in \mathbb{N}$$

as $m \rightarrow \infty$ the result that $By = Ax$ and this shows that $A \in (\int bv(u, w) : Y)$. This completes the proof. \square

Theorem 4.2. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $C = (c_{nk})$ are connected with the relation $c_{nk} = \bar{b}_{nk}$ for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y : \int bv(u, w))$ if and only if $C \in (Y : \ell_1)$.

Proof. Let $z = (z_k) \in Y$ and consider the following equality:

$$\sum_{k=1}^m c_{nk} z_k = \sum_{j=1}^n j \cdot u_n (w_j - w_{j+1}) \left(\sum_{k=1}^m a_{jk} z_k \right) \quad (24)$$

for all $m, n \in \mathbb{N}$. Equation (24) yields as $m \rightarrow \infty$ the result that $(Cz)_n = \{\Gamma(Az)\}_n$. Therefore, one can immediately observe from this that $Az \in \int bv(u, w)$ whenever $z \in Y$ if and only if $Cz \in \ell_1$ whenever $z \in Y$. \square

Theorem 4.3. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $D = (d_{nk})$ are connected with the relation

$$a_{nk} = \sum_{j=k}^{\infty} \frac{1}{j} \cdot (w_k - w_{k+1}) u_j d_{nj} \quad \text{or} \quad d_{nk} = \tilde{a}_{nk}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then $A \in (d(bv(u, w)) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{d(bv(u, w))\}^\beta$ for all $n \in \mathbb{N}$ and $D \in (\ell_1 : Y)$.

Theorem 4.4. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation $e_{nk} = \tilde{b}_{nk}$ for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y : d(bv(u, w)))$ if and only if $E \in (Y : \ell_1)$.*

5 Examples

Example 5.1. *The Euler sequence space e_∞^r is defined by $e_\infty^r = \{x \in \omega : \sup_{n \in \mathbb{N}} |\sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k x_k| < \infty\}$ ([2]). We consider the infinite matrix $A = (a_{nk})$ and define the matrix $F = (f_{nk})$ by*

$$f_{nk} = \sum_{j=0}^n \binom{n}{j} (1-r)^{n-j} r^j a_{jk} \quad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(\int bv(u, w) : e_\infty^r)$ in Theorem 4.1, then, we replace the entries of the matrix A by those of the matrix F .

Example 5.2. *Let $T_n = \sum_{k=0}^n t_k$ and $A = (a_{nk})$ be an infinite matrix. We define the matrix $H = (h_{nk})$ by*

$$h_{nk} = \frac{1}{T_n} \sum_{j=0}^n t_j a_{jk} \quad (k, n \in \mathbb{N}).$$

Then, the necessary and sufficient conditions in order for A belongs to the class $(\int bv(u, w) : r_\infty^t)$ are obtained from in Theorem 4.1 by replacing the entries of the matrix A by those of the matrix H ; where r_∞^t is the space of all sequences whose R^t -transforms is in the space ℓ_∞ [17].

Example 5.3. *In the space r_∞^t , if we take $t = e$, then, this space become to the Cesàro sequence space of non-absolute type X_∞ [20]. As a special case, Example 5.2 includes the characterization of class $(\int bv(u, w) : r_\infty^t)$.*

Example 5.4. *The Taylor sequence space t_∞^r is defined by $t_\infty^r = \{x \in \omega : \sup_{n \in \mathbb{N}} |\sum_{k=n}^\infty \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k| < \infty\}$ ([16]). We consider the infinite matrix $A = (a_{nk})$ and define the matrix $P = (p_{nk})$ by*

$$p_{nk} = \sum_{k=n}^\infty \binom{k}{n} (1-r)^{n+1} r^{k-n} a_{jk} \quad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(\int bv(u, w) : t_\infty^r)$ in Theorem 4.1, then, we replace the entries of the matrix A by those of the matrix P .

Similar to above examples, we can give necessary and sufficient conditions for the class $(d(bv(u, w)) : Y)$ in Theorem 4.3, where $Y \in \{e_\infty^r, r_\infty^t, X_\infty, t_\infty^r\}$.

If we take the spaces ℓ_∞ , c , c_0 , bs , cs and c_0s instead of X in Theorem 4.3, or Y in Theorem 4.1 we can write the following examples. Firstly, we give some conditions and following lemmas:

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} a_{nk} \right| < \infty, \quad (25)$$

$$\lim_k a_{nk} = 0 \text{ for each } n \in \mathbb{N}, \quad (26)$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k+1}) \right| < \infty, \quad (27)$$

$$\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k-1}) \right| < \infty \quad (28)$$

Lemma 5.5. *Consider that the $X \in \{\ell_\infty, c, c_0, bs, cs, c_0s\}$ and $Y \in \{\ell_1\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be read the following, from Table 2:*

7. (25)	8. (26), (27)	9. (28)	10. (27)
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From \rightarrow	ℓ_∞	c	c_0	bs	cs	c_0s
To \downarrow						
ℓ_1	7.	7.	7.	8.	9.	10.

Table 2

Example 5.6. *We choose $X \in \{\int bv(u, w)\}$ and $Y \in \{\ell_\infty, c, c_0\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be taken from the Table 3:*

- 1a.** (12) holds with \bar{a}_{nk} instead of a_{nk} .
- 2a.** (12) and (13) hold with \bar{a}_{nk} instead of a_{nk} .
- 3a.** (12) and (13) hold with $\alpha_k = 0$ as \bar{a}_{nk} instead of a_{nk} .
- 4a.** (15) holds with \bar{a}_{nk} instead of a_{nk} .
- 5a.** (15), (16) hold with \bar{a}_{nk} instead of a_{nk} .
- 6a.** (15), (17) hold with \bar{a}_{nk} instead of a_{nk} .

To \rightarrow	ℓ_∞	c	c_0	bs	cs	c_0s
From \downarrow						
$\int bv(u, w)$	1a.	2a.	3a.	4a.	5a.	6a.

Table 3

Example 5.7. *We choose $X \in \{d(bv(u, w))\}$ and $Y \in \{\ell_\infty, c, c_0, bs, cs, c_0s\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be taken from the Table 4:*

- 1b.** (12) holds with \tilde{a}_{nk} instead of a_{nk} .
- 2b.** (12) and (13) hold with \tilde{a}_{nk} instead of a_{nk} .
- 3b.** (12) and (13) hold with $\alpha_k = 0$ as \tilde{a}_{nk} instead of a_{nk} .
- 4b.** (15) holds with \tilde{a}_{nk} instead of a_{nk} .
- 5b.** (15), (16) hold with \tilde{a}_{nk} instead of a_{nk} .
- 6b.** (15), (17) hold with \tilde{a}_{nk} instead of a_{nk} .

To \rightarrow	ℓ_∞	c	c_0	bs	cs	c_0s
From \downarrow						
$d(bv(u, w))$	1b.	2b.	3b.	4b.	5b.	6b.

Table 4

Using the Lemma 5.5, we can give the Table 5 for $X \in \{\ell_\infty, c, c_0, bs, cs, c_0s\}$ and $Y \in \{\int bv(u, w)\}$ and Table 6 for $X \in \{\ell_\infty, c, c_0, bs, cs, c_0s\}$ and $Y \in \{d(bv(u, w))\}$ as follows:

- 7a.** (25) hold with \bar{b}_{nk} instead of a_{nk} .
- 8a.** (26) and (27) hold with \bar{b}_{nk} instead of a_{nk} .
- 9a.** (28) holds with \bar{b}_{nk} instead of a_{nk} .
- 10a.** (27) holds with \bar{b}_{nk} instead of a_{nk} .
- 7b.** (25) hold with \tilde{b}_{nk} instead of a_{nk} .
- 8b.** (26) and (27) hold with \tilde{b}_{nk} instead of a_{nk} .
- 9b.** (28) holds with \tilde{b}_{nk} instead of a_{nk} .
- 10b.** (27) holds with \tilde{b}_{nk} instead of a_{nk} .

From \rightarrow	ℓ_∞	c	c_0	bs	$cs,$	c_0s
To \downarrow						
$\int bv(u, w)$	7a.	7a.	7a.	8a.	9a.	10a.

Table 5

From \rightarrow	ℓ_∞	c	c_0	bs	$cs,$	c_0s
To \downarrow						
$d(bv(u, w))$	7b.	7b.	7b.	8b.	9b.	10b.

Table 6

6 Conclusion

The difference sequence spaces are given by Kızmaz [10]. If we choose the absolute summable sequence space and apply the difference operator to this space, we obtain the space of all sequences of bounded variation and denote by bv . The space bv_p consisting of all sequences whose differences are in the space ℓ_p . The space bv_p was introduced by Başar and Altay [1]. More recently, the sequence spaces bv and bv_p are studied in [1], [6], [8], [9], [13], [14], [15], [18].

Integrated and differentiated sequence spaces are introduced by [7]. Kirişci [14] have studied some properties of these spaces and defined the Riesz type integrated and differentiated sequence spaces [15]. In this work, we define the new integrated and differentiated sequence spaces. We also compute the dual spaces of these spaces which are help us in the characterization of matrix mappings. Therefore, we characterize the matrix classes. In last section, we give some examples related to the matrix transformations in the table form.

Competing Interests

The author declares that they have no competing interests.

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